The Interaction of Mathematics and Physics

Elliott Lieb
Princeton University

Royal Society
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Most of my scientific career was focused on the symbiotic relationship between mathematics and physics. I will give two examples: one, an application of mathematics to physics, and one where physics motivates a mathematical theorem.

EXAMPLE # 1, SQUARE ICE

The 3-color problem: Think of a very large $n \times n$ square lattice, with n^2 squares, i.e., a checkerboard that is very much larger than 8×8. We want to color the squares of the board in such a way that adjacent squares have different colors. If we have only 2 colors to play with (red and black) then it is easy to see that there are only two colorings, (red, black, red, ... or black, red, black, ...).

The picture changes if we have 3 colors on our palette! There are lots of colorings. How many? This is a problem in mathematical combinatorics, which I solved, and the answer, when n is very large, is:

$$\left[(4/3)^{3/2}\right]^{n^2} = [1.539600717839002\ldots]^{n^2},$$

and this number, $(4/3)^{3/2}$, is known as my **'square ice constant'**.

The solution to this problem quickly gave rise to many variations by many people and, it may be said, to a whole subfield of *statistical mechanics* known as *'vertex models'*.

What, one might ask, does this have to do with physics, ice, in particular, or with vertices?

This $n \times n$ lattice has n^2 squares, $2n^2$ edges and n^2 vertices (where edges meet). Each vertex has 4 edges. Let us assign an arrow (or direction) to each edge with the condition that at each vertex there are 2 arrows pointing into and 2 arrows pointing away from the vertex. This is called Pauling's ice rule.

How many arrow assignments are there? Answer: The same as the number of 3 colorings of the squares, $(4/3)^{3n^2/2}$. This explains the word 'vertex' model, but what about ice?

The vertices model the *oxygen atoms* in ice and the arrows indicate to which oxygen atom the *hydrogen atom* that sits between 2 oxygen atoms is closest. Pauling's ice rule is simply that, locally, we have ordinary $\mathrm{H_2O}$ molecules.

According to Boltzmann's famous formula of statistical mechanics, the entropy, S, of ice equals the logarithm of the number of configurations, namely,

$$S = (3n^2/2)\log(4/3)$$

. This is close to the measurement by Giauque and Stout in 1936!

EXAMPLE # 2, STABILITY OF MATTER

This subject occupied a sizeable part of my career. The question is the following:

Question: The 20th century began in *1897* with J.J. Thompson's discovery of the negatively charged electron! Ordinary matter is composed of electrons and positively charge atomic nuclei that attract each other. The question is: *Why do the electrons not fall into the nuclei, thereby reducing atoms and matter to insignificant size?*

Answer: Quantum mechanics and the *Uncertainty Principle.* Alas, this principle only leads to a non-zero size for matter, but it is not enough to explain why the size of a piece of matter is proportional to the amount of matter in the piece. Indeed, with gravitational forces the more matter there is the smaller is its size. But electric forces, thankfully, are better behaved and there is hope. Another principle, called the Pauli Exclusion Principle is also essential to understand the linear size of bulk matter. This linear law is called

Stability of Matter.

Problem: How do we know that Pauli's principle suffices to explain stability of matter? Is yet another principle needed? The problem I, and others, addressed is to find a mathematical proof of this sufficiency. This was accomplished by F. Dyson and A. Lenard in 1967, but their proof was quite complicated.

A more intuitive and easily understandable proof was needed.

The problem boiled down to studying the eigenvalues (i.e., quantum energy levels) of Schrödinger's equation. The uncertainty principle guarantees that the lowest eigenvalue is finite and not too negative (i.e., the electron does not fall into the atomic nucleus).

What turned out to be needed was a useful bound on how negative the *SUM of all the negative eigenvalues* can be. Not just the lowest one.

A bound of this kind was discovered by *Walter Thirring* (Univ. Vienna) and me in 1975. It is a theorem in pure mathematics motivated by a problem in physics. It became widely known since it has many applications in the theory of partial differential equations, differential geometry, turbulence and other areas.

THANKS FOR LISTENING!