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Teaching and learning geometry 11-19

Report of a Royal Society / Joint Mathematical Council working group

Teaching and learning geometry 11-19

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Preparation of this report

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Chairman's preface

Adrian Oldknow

*"About binomial theorem I'm teeming with a lot of news,
With many cheerful facts about the square of the hypotenuse"*
(WS Gilbert, *The Pirates of Penzance*)

The mathematical content for pupils following the National Curriculum in secondary schools in England is described under the headings of: *Number and algebra*; *Shape, space and measures* and *Handling data*. However the term 'numeracy' has become increasingly used in place of mathematics in relation to school education. This is an unfortunate practice since it downplays two areas, algebra and geometry, which are of major importance in school mathematics. The teaching of each of these aspects of mathematics has now been the subject of commissioned reports from the Royal Society and the Joint Mathematical Council of the United Kingdom, and the Qualifications and Curriculum Authority is currently engaged in a three year project on developing the teaching of both algebra and geometry.

A past President of the Royal Society, Sir Michael Atiyah, provided some succinct background to the development of algebra and geometry in a lecture given in Toronto in June 2000:

> *I want to talk now about a dichotomy in mathematics, which has been with us all the time, oscillating backwards and forwards... I refer to the dichotomy between geometry and algebra. Geometry and algebra are the two formal pillars of mathematics; they both are very ancient. Geometry goes back to the Greeks and before; algebra goes back to the Arabs and the Indians, so they have both been fundamental to mathematics, but they have had an uneasy relationship.*
> *Let me start with the history of the subject. Euclidean geometry is the prime example of a mathematical theory and it was firmly geometrical, until the introduction by Descartes of algebraic coordinates, in what we now call the Cartesian plane. That was an attempt to reduce geometrical thinking to algebraic manipulation.*
> (Reprinted in *Mathematics Today*, **37**(2), April 2001 46-53.)

At school level, algebra can seem quite abstract and cerebral. In Fitzgerald's studies for the Cockcroft committee it was algebra which was most frequently cited as the part of mathematics where adults remembered losing touch with mathematics. On the other hand, there are clear links in geometry to the world of our senses and experience. For example, we can easily perceive when objects are parallel, or perpendicular, or symmetrical - such as recognising when a minute adjustment is needed to the way a picture hangs. Sir Michael offers the following comments on our capacity to perceive, and its relationship with geometry:

> *Our brains have been constructed in such a way that they are extremely concerned with vision. Vision, I understand from friends who work in neurophysiology, uses up something like 80 or 90 percent of the cortex of the brain...*
> *Understanding, and making sense of, the world that we see is a very important part of our evolution. Therefore spatial intuition or spatial perception is an enormously powerful tool and that is why geometry is actually such a powerful part of mathematics - not only for things that are obviously geometrical, but even for things that are not. We try to put them into geometrical form because that enables us to use our intuition. Our intuition is our most powerful tool... I think it is very fundamental that the human mind has evolved with this enormous capacity to absorb a vast amount of information, by instantaneous visual action, and mathematics takes that and perfects it.*

Geometry is of far reaching importance beyond the worlds of professional mathematicians and of mathematics teaching. Geometry is frequently used to model what we call the 'real world' and has many applications in solving practical problems. (It is interesting to note that the French term for a surveyor is '*un expert géomètre*'.) Geometry is making contributions to many important scientific developments such as the Human Genome Project, Buckminster-Fullerene research, and whole-body tomography. Through media such as film, television and computer games we encounter computer generated geometric images of great complexity, and children and adults alike derive pleasure from creating designs and patterns exhibiting geometric forms.

So geometry is an important subject, with wide applications and a long history. It deals with matters we find attractive and for which we have a strong visual capacity. On the surface, then, it would appear that geometry should be one of the easiest branches of mathematics to teach. But this is not the case - neither in England nor in much of the developed world. This Royal Society / JMC study set out to identify why this is so.

Geometry is one of the oldest branches of mathematics - itself one of the oldest of mankind's intellectual studies. No wonder, then, that it suffers from an embarrassment of riches in terms of theories, results, techniques and applications. Many of these are well within the grasp of most, if not all, students in 11-19 education. We might refer to this, not unwelcome, problem as one of **abundance.** Clearly, then, choices have to be made on what material to include in the curriculum. At one extreme there is a danger of choosing eclectically from this abundance in a way that leads to the teaching of a lot of apparently unconnected 'bits'. At the other extreme there is a danger of developing a tightly organised body of knowledge which addresses only a very small part of geometry. Our challenge has been to combine breadth with both educational and mathematical coherence - a problem we refer to as **coherence**.

One of the less obvious difficulties in teaching geometry lies in the abstractions we make – we illustrate points and line segments through drawings and diagrams and yet neither object can be visible, except in our 'mind's eye'. We do not often choose to discuss such a difficult issue! Frequently however, teachers will draw rapid sketches purporting to represent objects in their own imagination which may actually not be recognised as such by their pupils.

The geometry of the ancient Greeks, as recorded by Euclid, was far more than a summary of known facts - it was an organised body of knowledge starting with a number of definitions and assumptions (axioms) which used logical deduction to establish a series of results in the form of theorems together with proofs. It is through the teaching of geometry that most pupils still encounter at least one theorem, that of Pythagoras, together with one or more proofs, and maybe some applications. My non-scientific guess is that most adults will remember the name Pythagoras, and probably that his theorem has to do with right angled triangles and words like 'hypotenuse', but that it would be extraordinary if they could remember a proof of the theorem. The role of proof, and the range of pupils for whom it is relevant, remains a major issue in geometry teaching. Despite the long tradition for the inclusion of geometrical proof in school curricula there is little evidence that we have developed effective methods for its teaching. Nevertheless, the working group supports the inclusion of proof in school geometry both because of its central role in mathematics, and as a contribution to developing more general skills of argument and criticism.

In order to address the issue of coherence the working group has followed on from a previous review of the geometry curriculum [Wynne Willson, 1977] and formulated a set of objectives for the teaching of geometry in the 21st Century. Against these objectives we have concluded that the geometrical content of the National Curriculum does provide a reasonable basis for the 11-16 curriculum, but needs strengthening in two main areas. These concern work in 3-dimensions, and in the educational application of Information and Communications Technology (ICT). Leaving the geometrical content relatively unchanged for now will allow scope for dealing with the issue which the working group has identified as by far the most important one for 11-16 geometry. That is to ensure that teachers have the knowledge, understanding, skills and resources to teach geometry in a way which genuinely captures pupils' interest and imagination, while developing their thinking and reasoning skills, their powers of visualisation, their ability to apply and model, and their understanding.

The 11-16 geometry curriculum in England continues to concentrate on techniques for working in 2 dimensions, such as the plane geometry derived from Euclid, together with elements of transformation, vector and coordinate geometry. Yet little of this finds its way into current AS/A-level specifications in mathematics, whose geometrical content has been drastically reduced over time. Similarly, the kind of geometry studied by mathematics undergraduates bears little resemblance to that studied either pre- or post-16. We refer to this issue as one of **progression.**

While the working group is optimistic about the possibility for significant improvement in teaching geometry 11-16, (which is not to underestimate the challenges to be addressed), it is far less sanguine about the state of geometry in 16-19 education. The geometrical content of the current AS/A-level specifications in pure mathematics is very small and offers little by way of progression from what has come before. But there is little point in advising content changes at this level when the whole basis of 16-19 qualifications in mathematics and all other subjects has just undergone a series of changes, the consequences of which have yet to be fully felt. Our view is that the general position of mathematics in 16-19 education needs a fundamental review before geometry can be accorded an acceptable place.

It is widely recognised that secondary schools have problems recruiting and retaining mathematics teachers. Many of those currently teaching mathematics in secondary schools are not mathematics graduates. Due to the problem of progression, it cannot be assumed that even trained mathematics graduates are adequately equipped to teach geometry in the way the working group envisages. To remedy this will require a substantial programme of well planned continuing professional development for teachers which improves both their subject knowledge in geometry and their approaches to teaching it. The current Key Stage 3 mathematics strategy provides substantial opportunities for the professional development of mathematics teachers in secondary schools and has the potential to make a valuable contribution to improvements in the teaching of geometry. But this alone will not be sufficient to improve teaching throughout the 11-19 sector.

Computer software, particularly that known as Dynamic Geometry Software, has the potential to make significant improvements in how geometry is learnt and taught. But such software is not widely available in school mathematics classrooms, as is the case with computing resources in general. In order for such resources to have maximum effect on improving the teaching and learning of geometry we need to find ways which allow talented teachers the time to develop a range of effective Information and Communication Technology based approaches. In addition to ICT, there is a need for a range of good materials to support the teaching of geometry in school. These, too, need to be carefully prepared and tried out, and that will also require time and effort.

The working group has been challenged to articulate its vision for geometry teaching. I believe that what we seek is a coherent, stimulating, rewarding and challenging geometry curriculum which is taught in a way which captures students' interest and imagination and which attracts them towards mathematics as a subject for further study. The achievement of our vision requires a significant improvement in the quality of teaching, and this has major consequences - both for the continuing professional development of teachers and for the provision of high quality supporting resources.

Summary

This report presents the findings of a broadly based working group established by the Royal Society and the Joint Mathematical Council to consider the teaching and learning of geometry in schools and colleges. The study was initiated following the publication of results of international educational comparisons, the 1999 revision of the National Curriculum for English schools 11-16, and at a time of several major policy initiatives in education.

The working group considered the rationale for a geometry curriculum, its possible content and issues concerned with its effective teaching. This report reflects its agreed views on the state of geometry teaching 11-19 and the major issues needing to be addressed to bring about improvements. It is supported by additional materials, some of which are printed here as appendices, and others of which are accessible from the Royal Society's website at www.royalsoc.ac.uk These additional materials are intended to help illustrate some of the points in the report, and to offer examples of approaches which might be taken by schools and colleges. They are sometimes attributed to an individual or groups of members and are then not claimed to represent the views of the whole group.

In order to help identify major issues raised, the report is structured around a number of agreed Key Principles. In the main body of the report these are presented together with explanations, supporting arguments and, where available, evidence. Additional information and exemplification is provided in the appendices. One or more recommendations are associated with each Key Principle.

Overall, for mathematics 11-16, we conclude that the geometrical content of the new National Curriculum, with a few adjustments, forms an appropriate basis for a good geometry education. In order for this to be achieved, however, considerable changes are needed in the way geometry is taught. It is vital that those working to improve mathematics education ensure that their work contributes significantly to improvements in geometry (as well as mathematics) teaching. Bringing about improvements in geometry teaching will require a significant commitment to a substantial programme of continuing professional development alongside the development of appropriate supporting materials.

For mathematics post-16 we conclude that there are insufficient opportunities for students to build on their 11-16 studies in geometry. Those concerned with curriculum design need to review the structure of post-16 qualifications in mathematics to ensure they provide improved opportunities for students to continue to study geometry. The provision of challenging and interesting geometry should help make mathematics a more attractive subject of study for more students. This in turn would contribute to overcoming the current shortage of those with good mathematical skills.

Key Principles

Key Principle 1: Geometry should form a significant component of the mathematics curriculum for all students from 11 to 19.

Key Principle 2: Any choice of curriculum should be underpinned by a rationale.

Key Principle 3: The geometry curriculum should maintain breadth, depth and balance, and be consistent with Key Principle 2 and the objectives in Recommendation 3.

Key Principle 4: Geometry should be given a higher status, together with a fair share of the teaching time available for mathematics.

Key Principle 5: Students in 16-19 education should have the opportunity to continue further their studies in geometry.

Key Principle 6: The assessment framework for the curriculum should be designed to ensure that the full range of students' geometrical knowledge, skills and understanding are given credit.

Key Principle 7: The most significant contribution to improvements in geometry teaching will be made by the development of good models of pedagogy, supported by carefully designed activities and resources, which are disseminated effectively and coherently to and by teachers.

Key Principle 8: It is a matter of national importance that as many of our students as possible fully develop their mathematical potential. Geometry, with its distinctive appeal, should make mathematics attractive to a wider range of students.

Recommendations

Recommendation 1: We recommend that curriculum and assessment specifications be reviewed to ensure that geometry forms a significant component of the mathematics curriculum for all students from 11 to 19.

Recommendation 2: We recommend that the title of the attainment target Ma3 of the National Curriculum be changed from 'Shape, space and measures' to 'Geometry'.

Recommendation 3: We recommend that the geometry curriculum be chosen and taught in such a way as to achieve the following objectives:

a) to develop spatial awareness, geometrical intuition and the ability to visualise;
b) to provide a breadth of geometrical experiences in 2- and 3-dimensions;
c) to develop knowledge and understanding of and the ability to use geometrical properties and theorems;
d) to encourage the development and use of conjecture, deductive reasoning and proof;
e) to develop skills of applying geometry through problem solving and modelling in real world contexts;
f) to develop useful Information & Communication Technology (ICT) skills in specifically geometrical contexts;
g) to engender a positive attitude to mathematics; and
h) to develop an awareness of the historical and cultural heritage of geometry in society, and of the contemporary applications of geometry.

Recommendation 4: We recommend that the current geometrical content of the English secondary school mathematics National Curriculum be regarded as a reasonable basis for an appropriate and rewarding geometry education for all pupils.

Recommendation 5: We recommend that the mathematics curriculum be developed to encourage students to work investigatively, demonstrate creativity and make discoveries in geometrical contexts so that students develop their powers of spatial thinking, visualisation and geometrical reasoning.

Recommendation 6: We recommend that the mathematics curriculum be developed in ways which recognise the important position of theorems and proofs within mathematics and use the study of geometry to encourage the development of logical argument appropriate to the age and attainment of the student.

Recommendation 7: We recommend that the mathematics curriculum be developed to provide ample opportunities for students to use geometry for practical problem solving through mathematical modelling in both 2- and 3-dimensions.

Recommendation 8: We recommend that the geometry curriculum be developed to give greater emphasis to work in 3-dimensions and to make better use of Information and Communication Technology (ICT).

Recommendation 9: We recommend that the use of the word 'numeracy' in government publications and announcements be replaced by 'mathematics' to ensure that geometry is accorded its rightful position.

Recommendation 10: We recommend that geometry should occupy 25% - 30% of the teaching time, and hence a similar proportion of the assessment weighting, in the 11-16 mathematics National Curriculum.

Recommendation 11: We recommend that the total time allocated to mathematics 11-16 be monitored to ensure students spend at least 3 hours a week on mathematics, so that sufficient time is given to the teaching of geometry, and to other aspects of mathematics.

Recommendation 12: We recommend that a fundamental review be made of all 16-19 mathematics provision. This should include considering how:

a) the structure and content of the current AS/A-level Mathematics and Further Mathematics specifications can better meet the needs of students and include a greater emphasis on geometry; and
b) other post-16 mathematics qualifications, such as Free Standing Mathematics Units (FSMUs) and AS-level Use of Mathematics, can enable students to have the opportunity to continue their study of geometry.

Recommendation 13: We recommend that in the 16-19 curriculum the key skill, 'Application of Number', be re-titled 'Application of Mathematics' and that the range of qualifying mathematical studies be broadened so that students continue their study of geometry.

Recommendation 14: We recommend that a review be made of the methods of assessment and examination used in mathematics at Key Stage 3, at GCSE and in post-16 qualifications to ensure that appropriate credit is given for the attainment of specific geometrical objectives.

Recommendation 15: We recommend that the relevant government agencies work together, with bodies such as the mathematics professional associations represented on JMC, to provide a coherent framework for supporting the development of teaching and learning in geometry. This will involve:

a) the recognition and development of good practice in geometry teaching through pilot studies and research;
b) the design of programmes of continuing professional development and initial teacher education;
c) the production of supporting materials; and
d) the establishment of mechanisms to provide supporting resources, including ICT.

Recommendation 16: We recommend, in terms of mathematics in general, that:

a) better publicity and information be provided to schools, students and parents about the career opportunities afforded by studying mathematics; and
b) ways be sought to encourage schools and colleges to attract more students to study mathematics post-16, particularly at A-level.

1 Introduction

This report presents the findings of a working group established by the Royal Society and the Joint Mathematical Council (JMC) to consider the teaching and learning of geometry in schools and colleges. The working group, chaired by Professor Adrian Oldknow, met fourteen times between February 2000 and May 2001. The membership of the working group is given at the front of this report and its terms of reference can be found in Appendix 1.

The study was initiated following publication of the results of international educational comparisons, the 1999 revision of the National Curriculum for English schools 11-16 and at a time of several major policy initiatives in education. Some of this background is set out in Appendices 2 and 3.

Membership of the group was carefully chosen to include those with experience in: (a) teaching mathematics in state and independent schools and colleges, in initial teacher education and in higher education; (b) conducting research in mathematics (including geometry) and in mathematics education; (c) applying mathematics (including geometry) in disciplines such as science, engineering, IT and finance, and; (d) planning and implementing mathematics curricula in Local Education Authorities (LEAs) and government agencies. A variety of groups have expectations of the mathematics curriculum and its geometrical content; some of these are considered in Appendix 4.

2 Geometry and its teaching and learning

Geometry is one of the longest established branches of mathematics. It has an extensive range of applications and we give some selective historical and cultural background in Appendix 5. Geometry has been accorded a central place in mathematical education in Western culture for a considerable period of time. One of the major achievements of classical geometry was the systematic collection by Euclid of the geometrical knowledge of the ancient Greeks. This has, until comparatively recently, formed the basis for much of the geometry taught in schools.

During a period of educational reforms in mathematics in the 1950s and 1960s some new syllabuses (sometimes called 'the new maths') were developed where the emphasis was on formal structures which were predominantly algebraic. At the same time, the range of approaches to geometry was broadened from its traditional Euclidean base (which was reduced in depth) to include the use of transformations, vectors, matrices and some topology.

In recent years many countries have been reviewing the aims, content and approach of their geometry curricula. The 1995 study by the International Commission on Mathematics Instruction (ICMI) [Mammana and Villani, 1998] revealed that no clear consensus was emerging about the outcome of these reviews. The small scale research study into the geometry curricula of a number of countries commissioned in 2000 by the Qualifications and Curriculum Authority (QCA) for England confirmed this.

Against this background the working group considered the rationale for a geometry curriculum, its possible content and issues concerned with its effective teaching. Our report sets out a number of recommendations on issues where the working group reached a consensus view. There are some matters on which the working group did not reach a conclusion, and which others may wish to pursue further. There are also some matters which the working group did not address. In order to help identify major issues raised, the report is structured around a number of agreed Key Principles. These are presented together with explanations, supporting arguments and, where available, evidence. Additional information and exemplification are provided in appendices and on the Royal Society website at www.royalsoc.ac.uk One or more recommendations are associated with each Key Principle.

3 The place of geometry in the curriculum

Key Principle 1: Geometry should form a significant component of the mathematics curriculum for all students from 11 to 19.

This is a simple proposition to express yet it has many facets. First we consider some issues about the role of geometry in education. Then we consider the relation of geometry to other aspects of the mathematics curriculum. We review some of the problems associated with teaching aspects of geometry and pave the way for other key principles which stem from this.

A valid case for the study of geometry may be made on several grounds. Geometry is a central part of mathematics, and geometrical thinking is a fundamental way to engage with mathematics. Geometry can be used to develop students' spatial awareness, intuition and visualisation. It can also be used to solve practical problems. There are many applications of geometry relevant to employment and everyday life. Other subjects in the curriculum, such as science and technology, make use of geometrical ideas and techniques. Geometry is well established in our culture and has an interesting history of its own. It has an important place in the development of aesthetics and design. It can be used to encourage the development and use of conjecture, deductive reasoning and proof. Geometry can also be used to lay foundations for further studies in mathematics.

It is our view that all of these grounds, which have often been cited in the past, remain valid reasons for the inclusion of geometry as a significant part of the current curriculum. There are additional grounds that reflect recent changes in our society.

The rapid development in a range of technologies means that citizens now and in the future will interact with a variety of forms of displayed images. These may be required by their work, be needed in order to exchange information or just be associated with leisure. A case can thus be made that geometry has a role to play within the development of citizenship in enabling students to interpret, manipulate, control and create such images.

In recent years there has been a major shift in the UK economy from manufacture to services. Associated with this has been a marked increase in demand for those with good skills in flexible thinking and the use of Information and Communication Technology (ICT), together with the ability to apply mathematics (inadequately referred to as 'numeracy' skills). A direct consequence has been the much publicised problem in recruiting and retaining mathematics teachers. In order to fulfil the skills needs of industry, commerce and the professions - including teaching - we need to encourage more students to engage positively with mathematics and to choose to continue their studies in it, or related disciplines. We believe that geometry is a subject of mathematical study which has its own appeal and satisfaction and which, well taught, could encourage more students to continue with the study of mathematics beyond 16.

Breadth of study in geometry needs to be provided to meet the demands outlined above. To ensure students also receive appropriate intellectual challenges and stimuli it is important to provide depth in a number of topics. The challenge, of course, is to do both within a fair share of the time which should be allocated to mathematics teaching.

We conclude this first Key Principle with two recommendations. We believe that geometry has declined in status within the English mathematics curriculum and that this needs to be redressed. It should not be the "*subject which dare not speak its name*".

Recommendation 1:

We recommend that curriculum and assessment specifications be reviewed to ensure that geometry forms a significant component of the mathematics curriculum for all students from 11 to 19.

Recommendation 2:

We recommend that the title of the attainment target Ma3 of the National Curriculum be changed from 'Shape, space and measures' to 'Geometry'.

4 The 11-16 curriculum

Key Principle 2: Any choice of curriculum should be underpinned by a rationale.

Here we work towards defining a set of objectives against which to evaluate the geometrical content of a curriculum. First we summarise the current position regarding the 11-16 curriculum for the maintained sector in England.

The English educational system, centrally administered by the Department for Education and Skills[1] (DfES), is organised around a number of relatively autonomous agencies and units. These include the Qualifications and Curriculum Authority (QCA), the Office for Standards in Education (Ofsted), the Teacher Training Agency (TTA) and the British Educational and Communications Technology Agency (BECTa).

Schools and colleges have already had to adapt to considerable changes in very short time scales. So, rather than attempting to develop a geometry curriculum from first principles, we have chosen to review the current curriculum.

First we consider issues concerned with the teaching of mathematics in England in secondary schools to pupils aged 11-16. The QCA published a revised version of the National Curriculum for England in 1999 for implementation in schools and colleges from September 2000. The mathematics curriculum at Key Stages 3 (ages 11-14) and 4 (ages 14-16) differs from the earlier version in a number of respects. In particular the new version details the curriculum separately for each Key Stage, whereas the earlier version combined Key Stages 3 and 4. It also divides the Key Stage 4 curriculum into two programmes of study called 'mathematics foundation' and 'mathematics higher'. The geometrical content of the new curriculum is described within the section Ma3 *Shape, space and measures*. It is described in much greater detail than in the previous version. It has been suggested that the earlier version gave scope for teachers to address the items of geometrical content found in the 1999 version. However it is our experience that some significant aspects of geometry in the new version, particularly in the higher programme at Key Stage 4, are not currently taught extensively in secondary schools.

The working group considered the way in which the 11-16 mathematics curriculum is presented, and currently examined. At Key Stage 3 there is a single curriculum for all pupils. At Key Stage 4 it is divided into two. It is anticipated that roughly half of pupils will not study many of the additional aspects of mathematics contained only in the higher programme of study. Currently the examinations for mathematics in the General Certificate of Secondary Education (GCSE) are set in three tiers: foundation, intermediate and higher. The introduction of separate programmes of study alongside the use of three examination tiers raises a number of issues. The working group chose not to consider alternative ways of packaging the curriculum as these structures have implications for the whole mathematics curriculum, not just geometry.

The original National Curriculum has gone through two sets of revisions; neither of these has provided a rationale for the content of the mathematics curriculum. In our discussions, we identified a clear need to provide a set of objectives against which curriculum content should be evaluated and which we now provide in the form of a recommendation to improve the focus, coherence and relevance of geometry teaching.

Recommendation 3:

We recommend that the geometry curriculum be chosen and taught in such a way as to achieve the following objectives:

a) to develop spatial awareness, geometrical intuition and the ability to visualise;
b) to provide a breadth of geometrical experiences in 2- and 3-dimensions;
c) to develop knowledge and understanding of, and the ability to use, geometrical properties and theorems;
d) to encourage the development and use of conjecture, deductive reasoning and proof;
e) to develop skills of applying geometry through problem solving and modelling in real world contexts;
f) to develop useful Information & Communication Technology (ICT) skills in specifically geometrical contexts;
g) to engender a positive attitude to mathematics; and
h) to develop an awareness of the historical and cultural heritage of geometry in society, and of the contemporary applications of geometry.

From the analysis of the current geometry curriculum a number of questions emerged to which the working group has worked to find answers:

- How should the geometrical content be determined?
- Does the content of the revised Ma3 curriculum form

[1] The DfES was created in June 2001. Before this, the English education system was administered by the Department for Education and Employment (DfEE)

an appropriate basis for the teaching of geometry in secondary schools at Key Stages 3 and 4?

- If not, how should it be modified?
- What suppositions are made about the time available for teaching mathematics, and its geometry component, and are these acceptable?
- What issues need to be addressed if it is to be taught effectively?
- How do assessment procedures impact on teaching?
- What are the implications for teaching geometry pre-11 and post-16?

There is no requirement for the development of the National Curriculum to be based on evaluated field trials and experiments to test feasibility. Nor are such developments necessarily linked with any associated professional development for teachers, or with any development of appropriate teaching materials or assessment. Thus, in the absence of evidence, it has to be a matter of judgement whether the geometry selected for inclusion in the content of the National Curriculum defines an attainable curriculum.

The Ma3: *Shape, space and measures* component of the 1999 National Curriculum certainly exhibits a breadth of study in geometry (see Appendix 6). It is the view of the working group, led by the experienced school teachers amongst us, that given the right circumstances it can provide an appropriate, interesting and attainable curriculum. We shall discuss what we mean by the right circumstances later in this report.

Before considering the curriculum content in greater detail we consider some recent changes in the way the mathematics curriculum is implemented and developed. The first of these is the model followed by the National Numeracy Strategy (NNS) in primary schools, which is now being extended to mathematics at Key Stage 3 in secondary and middle schools. The second is the 3 year project concerned with the teaching of algebra and geometry now being conducted by the QCA.

The National Numeracy Strategy is managed by the DfES's Standards and Effectiveness Unit (SEU). It has developed a year by year framework for teaching Key Stages 1 and 2 of the mathematics National Curriculum. This is based on work carried out in over 200 pilot schools. Associated with the detailed teaching schemes has been a large scale professional development exercise involving LEAs, headteachers, mathematics coordinators, classroom teachers, governors etc. It has thus served as a national medium for the interpretation and implementation of the established curriculum. The government has extended the work of the Strategy first into Year 7 (the year of entry to most secondary schools), and more recently into the whole of Key Stage 3. The Key Stage 3 mathematics strategy comes into national effect in September 2001 after a short pilot stage. Apart from the way it is being introduced, there are other differences between the primary and secondary strategies, among the most pressing of which is the current shortage of qualified mathematics teachers in secondary schools. We welcome the opportunities for the improvement in mathematics teaching in secondary schools which this large scale development has the potential to stimulate. In the latter stages of our work an observer from the Key Stage 3 mathematics strategy joined the working group in order to ensure better linkage between our conclusions and recommendations and the way the Key Stage 3 strategy will implement the teaching of Ma3. Brief examples from the current framework for mathematics in Years 7, 8 and 9 appear in Appendix 7, and there are links to the full document on the Royal Society website at www.royalsoc.ac.uk

An earlier working group of the Royal Society and JMC produced a report on the teaching of algebra pre-19 [Royal Society / JMC 1996]. The DfES is now supporting a 3 year study by the QCA into the teaching of Algebra and Geometry. This has already commissioned international studies into the teaching of those aspects of mathematics. See also Appendix 2 for a brief discussion of international trends. We welcome the opportunity which this new project offers to implement our recommendations for the teaching of geometry. We also welcome the extended time scale for this project.

Recommendation 4:

We recommend that the current geometrical content of the English secondary school mathematics National Curriculum be regarded as a reasonable basis for an appropriate and rewarding geometry education for all pupils.

5 The development of the curriculum

Key Principle 3: The geometry curriculum should maintain breadth, depth and balance, and be consistent with Key Principle 2 and the objectives in Recommendation 3 and Appendix 11.

In reviewing the curriculum we paid particular attention to a number of important aspects of geometry related to our recommended set of objectives.

Through tackling, and solving, problems in geometry (both closed and open ended) pupils can develop 'thinking skills' of reasoning, enquiry (which includes problem posing and conjecturing) and creativity. They can also develop their geometrical intuition and extend their powers of visualisation and spatial thinking. These aspects are considered further in Appendix 8.

An important aspect of geometry is concerned with the development of deductive reasoning and proof. Of course proof is not confined to geometry alone, and there can be interactions, such as algebraic results proved geometrically and vice versa. However the use of geometry as a vehicle for the development of the understanding and use of deductive reasoning has received relatively little emphasis in the English school curriculum over the last 30 years.

For a variety of reasons, the whole issue of proof within school geometry has become emotive. In some minds it is associated with a particular style of teaching and examining sometimes pejoratively, and erroneously, associated with the name Euclid. In others, it is regarded as the essential difference between mathematics and the experimental sciences, and as an essential tool for the further study of mathematics. The working group has had many interesting discussions about the place of geometrical proofs within mathematics, particularly at Key Stages 3 and 4. We have also received advice from individuals and bodies representing many shades of opinion - with a strong representation in favour of geometrical proof from correspondents in Higher Education and from some school teachers. We have concluded that it is important for all students to encounter proof during their study of geometry, while also recognising that some aspects of proof may be more accessible in other mathematical contexts. For a discussion of what we mean by proof see Appendix 9.

There is no suggestion here to attempt an axiomatic approach to school geometry. Indeed we note that such attempts have been made, unsuccessfully, in the past. Rather we are arguing for the use of logical argument, which builds upon what is already known by the pupil in order to demonstrate the truth of some geometrical result, possibly one conjectured by the pupil after conducting a well chosen experiment. The results concerned (i.e. the theorems) should be chosen as far as possible to be useful, interesting and/or surprising. The level of sophistication expected in the logical argument will depend upon the age and ability of the pupil concerned, and the proof produced might equally be called an 'explanation' or 'justification' or 'reason' for the result. Many pupils may never reach the level of providing formal proofs of results (although the more able should), but all should understand deductive reasoning and that it is more than simply stating a belief or checking that the result is valid in many specific cases. Encouraging pupils to be critical of their own, and their peers', explanations will help them develop the sophistication and rigour of their arguments. The emphasis at all times should be on understanding, and analysing a proof of a standard theorem has a positive role in understanding too. Without doubt, the end result of a proof at this level should be an understanding of why the result is true, not simply that a formal argument proves it. However, we accept that it is not an easy matter to determine how to achieve this with each pupil and each result and that a careful choice of approach will be needed. Some examples of proof in a variety of areas and styles appear in Appendices 9 and 11.

We are aware that there are considerable difficulties to be overcome in achieving our objective "*to encourage the development and use of conjecture, deductive reasoning and proof*" (Recommendation 3d). We do not have a successful experience base to fall back on, nor have we found that other countries have positive lessons to offer. We have found that many teachers currently in post or in training do not have experience in using geometrical reasoning themselves. We consider the implications of these issues in Key Principle 7 below.

Mathematical reasoning is one strand of a fundamental, and unusual, area of the mathematics National Curriculum called *Using and Applying Mathematics*. We now consider its other strands which relate to communication and problem solving. In previous versions of the National Curriculum this area was described as a separate component, Ma1. In the current version it has been integrated within each of the other three components, including Ma3 *Shape, space and measures*. The working group accepts that it is important for all students to appreciate the power of mathematics in the way it is applied in modelling important phenomena and solving practical problems - and that this applies equally in geometry as in other areas. We have concluded that it is important for all students to experience the applicability of geometry through engaging in mathematical modelling in 2- and 3-dimensions. Geometrical ideas are used in many models which pupils will encounter and use in the future. We are aware of the possibly confusing nature of the word 'model' in this context. By a 'mathematical model' we mean a representation through the language of mathematics of a real world problem. 'Modelling' is the process of translation into mathematics, usually involving simplification and idealisation. This modern use of the word recognises that this process of translation is itself a skill to be learned. It also recognises that the resulting mathematics will never be a perfect description of the original problem. We give some examples of the ways in which geometry can be used in school to model familiar situations in Appendix 10.

Another important feature of the geometry curriculum is that it provides opportunities for pupils to draw sketches, diagrams and accurate representations. We give just a few examples. Pupils can learn to use the properties of figures, such as isosceles triangles, rhombuses and kites, to develop mathematically exact constructions on paper with straight-edge and compass. They can explore whether sets of the same figures can be arranged to tile the plane. They can explore and apply properties of standard figures, as well as constructions, on computers using suitable geometry software. They can produce plane sections of 3-D objects to apply their knowledge of 2-D figures in solving problems in 3-D. They can sketch perspective drawings of 3-D objects from different viewpoints. They can make nets from which to construct 3-D solids.

The section on Transforming Secondary Education in the Green Paper, '*Schools, Building on Success*' of February 2001 includes the following:

> *4.29 The goals of our Key Stage 3 strategy are to ensure that by age 14, the vast majority of pupils have: .... Learnt how to reason, to think logically and creatively and to take increasing responsibility for their own learning.*

Responsibility for the development of pupils' thinking skills now comes within the 'Teaching and Learning in the Foundation Subjects' component of the government's Key Stage 3 strategy. The working group welcomes this recognition of the importance of thinking skills and recommends that the national Key Stage 3 strategy makes use of the geometry component of the mathematics curriculum for the development of such skills.

So, consistent with our stated objectives, the working group advocates striking a balance between the creative, deductive and applicable aspects of geometry.

Recommendation 5:

We recommend that the mathematics curriculum be developed to encourage students to work investigatively, demonstrate creativity and make discoveries in geometrical contexts so that students develop their powers of spatial thinking, visualisation and geometrical reasoning.

Recommendation 6:

We recommend that the mathematics curriculum be developed in ways which recognise the important position of theorems and proofs within mathematics and use the study of geometry to encourage the development of logical argument appropriate to the age and attainment of the student.

Recommendation 7:

We recommend that the mathematics curriculum be developed to provide ample opportunities for students to use geometry for practical problem solving through mathematical modelling in both 2- and 3-dimensions.

We now consider what might be missing from the current curriculum. The first matter we identified is the need for much greater attention to 3-D geometry at each stage of the curriculum for all pupils whatever their ability. It is simplistic just to note that we live in a 3-D world and need to be able to develop the geometrical skills to represent 3-D objects and to solve problems involving them. Clearly 3-D modelling is of great importance in a wide range of disciplines, such as science, engineering and design. We now come into contact with a much wider range of 2-D representations of 3-D objects than was previously the case. Spatial awareness, powers of visualisation and realistic means of applying geometry cannot be developed successfully without paying greater attention to work in 3-D. So we propose that in the 11-16 curriculum students should extend their understanding, skills and knowledge of geometry in the plane to solve problems in 3-D. Of course, some 3-D work relies on 2-D results which will need to be established first.

The revision of the National Curriculum by QCA in 1999 gave the opportunity for greater exemplification of the ways in which Information and Communication Technology impacts on many subjects and their teaching. Yet there is very little specific reference to the use of ICT in the mathematics National Curriculum in general, and in geometry in particular. Geometrical software is now widely used in, for example, engineering and design. Through government and commercial initiatives many secondary schools and colleges have acquired powerful Computer Aided Design and Computer Aided Manufacture (CADCAM) packages for use in teaching Design and Technology. By contrast relatively few schools have access to software for teaching geometry in mathematics. Yet by using such software in appropriate ways, pupils can apply their ICT skills to increase their knowledge and understanding of geometry. The software also provides them with the opportunity to acquire and practise geometrical skills. Opportunities occur when pupils create, analyse and interpret dynamic spatial images; make and test conjectures about geometrical relationships that they can manipulate; and record and present the results of their investigations.

As with any approach to teaching, the educational use of ICT needs to be well thought through and carefully planned. The TTA has produced documentation to accompany the current programme of lottery funded ICT training for all teachers in which it emphasises the importance of a critical approach to the use of ICT. This expects teachers to know where, when and how to apply ICT to enhance the teaching and learning of their subjects. This advice is particularly important in geometry where a variety of approaches is needed including mental, practical, and ICT enhanced work. Increasingly powerful software is becoming available in education, such as that designed for simulations in science and geography, much of which relies on sophisticated mathematical algorithms. Pupils and teachers in all subjects need to be cautious about accepting computer produced results without question, and mathematics is probably the subject best placed in the curriculum in which to engender a critical approach. In teaching geometry, caution is particularly needed to avoid making assertions based solely on computational illustrations.

Thus the working group would like to see further development of the curriculum with respect to work in 3-D and the use of ICT. Appendix 11 on 3-D geometry gives examples of five topics that are suitable for schools. We recognise that this will have implications for resources, materials, assessment and teachers' professional development, as will the **effective** teaching of proof, modelling, problem solving and other aspects of geometry. In many respects we need to develop a completely new pedagogy in geometry. We consider such issues further below. We also recognise that we are advocating an extension of the current curriculum, even before it has been fully implemented. Conscious of the potential criticism for proposing to extend an already crowded curriculum we address the issue of time allocation for mathematics below. Experienced teachers have developed their own mechanisms for setting out the curriculum in such a way that links can be made and time used most effectively. In Appendix 12 we include an extract from a possible framework for the extended geometry curriculum devised by some members of our working group.

Recommendation 8:

We recommend that the geometry curriculum be developed to give greater emphasis to work in 3-dimensions and to make better use of Information and Communication Technology (ICT).

6 Status and allocation of time to geometry

Key Principle 4: Geometry should be given a higher status, together with a fair share of the teaching time available for mathematics.

Recently there has been a tendency to replace the word 'mathematics' with 'numeracy', as if the two were equivalent. This has sent out mixed messages about the relative importance of different aspects of the mathematics curriculum. For example, within the pages of the DfES *The Standards Site* on the Internet (www.standards.dfee.gov.uk/numeracy/) the following description of the National Numeracy Strategy may be found:

> *Framework for teaching mathematics*
> *The Numeracy Framework helps teachers raise numeracy standards nationwide by providing them with a set of yearly teaching programmes, key objectives and a planning grid.*

Similarly the introduction to the government Green Paper, *Schools: Building on Success*, published in February 2001, contains the following:

> *Every secondary age pupil must be competent in the basics of literacy, numeracy and ICT and experience a broad curriculum beyond.*

We are concerned that this concentration on numeracy should not result in the sidelining of geometry.

Recommendation 9:

We recommend that the use of the word 'numeracy' in government publications and announcements be replaced by 'mathematics' to ensure that geometry is accorded its rightful position.

While accepting that the area called Ma2 *Number and algebra* in the secondary school curriculum should have the greatest amount of teaching time, we regard 25% of the available mathematics time as the minimum necessary for the teaching of geometry in Ma3. We are concerned about reports from some secondary schools that there has been an erosion in the total time available for teaching mathematics, particularly in Key Stage 3 - perhaps exacerbated by teacher shortages. Primary schools following the NNS have a daily mathematics lesson which, by the end of Key Stage 2, lasts one hour. Secondary schools following the new Key Stage 3 strategy have been given guidelines for the time to be allocated to mathematics - at least 3 hours per week. Members of the working group have also expressed the view that if mathematics is to have parity of esteem with the other core subjects of science and English then it should be available as a double award at GCSE.

We do not have specific proposals to make about the teaching of geometry in primary schools. The National Curriculum at both Key Stage1 (pupils aged 5-7) and Key Stage 2 (pupils aged 7-11) has the Ma3 *Shape, space and measures* component. The NNS's *Framework for teaching mathematics from Reception to Year 6* provides an interpretation of this within the framework of the daily mathematics lesson. Provided that this curriculum is effectively implemented, then pupils transferring from primary schools to Year 7 in secondary schools should have a suitable basis on which to develop their study of geometry.

The working group is aware that the effective teaching of the secondary school geometry curriculum which it advocates is likely to require rather more time for geometry than is currently normally the case. The renaming of Ma3 to 'Geometry' should imply that the work on non-geometrical measures, such as time and speed, is relocated in Ma2. Some aspects of Ma2 *Number and Algebra* could be developed within geometrical contexts, such as Pythagoras's Theorem.

Questions have been raised about the time, and assessment, allocation to Ma4 *Handling Data*, and even as to whether it should be part of the mathematics curriculum at all. However we do not wish to make any recommendations in respect of the content of this, or other, parts of the mathematics curriculum. That is not to duck the issue but to record that it is for others to assess the strength of our claims for geometry against those of other parts of the curriculum. We have already pointed to the lack of a sound experience base for an appropriate pedagogy for significant aspects of the geometry curriculum, for which there is an urgent need. It could well be that with the right approach, supported by appropriate materials and resources including ICT, the teaching of geometry could also be made more efficient. In summary we believe that a broad, coherent and demanding geometry curriculum can be effectively taught within a fair and reasonable time allocation. This may require some review of the balance between the components of the mathematics curriculum. It will certainly require the development of more effective and efficient teaching approaches.

Recommendation 10:

We recommend that geometry should occupy 25% - 30% of the teaching time, and hence a similar proportion of the assessment weighting, in the 11-16 mathematics National Curriculum.

Recommendation 11:

We recommend that the total time allocated to mathematics 11-16 be monitored to ensure students spend at least 3 hours a week on mathematics, so that sufficient time is given to the teaching of geometry, and to other aspects of mathematics.

7 Geometry 16-19

Key Principle 5: Students in 16-19 education should have the opportunity to continue further their studies in geometry.

The government has implemented reforms in the post-16 sector called 'Curriculum 2000'. Students are now encouraged to follow a programme of study which includes key skills, among which is the 'Application of Number'. For most students this will be the only course of mathematics they study post-16. Consistent with our first Key Principle, we propose that its title should be changed and its content extended so that students study material from a wider range of topics in mathematics, including geometry. There should be more compulsory elements of geometry which are assessed through tests, and which make explicit the opportunities to develop geometrical ideas in greater depth for inclusion in the portfolio.

The QCA have also recently revised their criteria for the mathematics General Certificate of Education Advanced Subsidiary and Advanced Level (AS- and A-level). Awarding bodies have now produced specifications that are being taught for the first time in the current academic year. The geometry in the compulsory part (core) of pure mathematics for A-level consists of a very small amount of coordinate geometry (lines and circles), some trigonometry and some elementary work with vectors. Within the current framework there is little scope for more geometry in the core, but more use could be made of geometrical contexts, say in the application of calculus. The working group doubts that the current geometrical content of A-level mathematics forms a suitable foundation for those students who go on to study science or engineering. In particular there should be greater emphasis on work in 3-D.

The working group has discussed the possibility of introducing one or more optional modules at AS- and A-level, outside the core of pure mathematics, which could include extensions in geometry. Potential drawbacks of such a solution would be the increase in the variety of routes to an award - leading to problems of comparability of standards, and also a greater variety of mathematical backgrounds of students taking the same course in higher education. We do not have an instant solution to propose with regard to improving the geometrical content of AS- and A-level mathematics in their current format. When a fundamental review of these qualifications takes place the working group recommends that careful consideration be given to extending the amount of geometry in the A-level core. Candidates for an extension to such content include plane curves, such as conics, further vector geometry and a greater emphasis on parametric representations.

We would expect a greater emphasis to be given to the important role of coordinate geometry as a link between algebra, graphs and functions, and calculus.

More generally there is reason to believe that the existing choice of optional modules (mainly in mechanics or statistics) does not meet the needs or interests of all potential candidates for A-level mathematics - such as those with an interest in aesthetics, or an intention to pursue careers in the IT industry. Currently about 60 000 of the c.230 000 A-level candidates enter for A-level mathematics. The uptake in mathematics at this level might be increased if there was a wider choice of modules, one or more of which included interesting geometry. We understand that evidence is beginning to emerge that the number of students taking AS-level mathematics as a fourth subject in the lower sixth-form (Year 12) and then not choosing to progress to A-level mathematics in the following year is higher than anticipated. However the working group is aware that much of this is speculation, and so we recommend that research be undertaken into the mathematical needs and interests of post-16 students, and the implication for the curriculum. Our conclusion is that both the structure and the content of AS- and A-level mathematics are in need of a fundamental review.

Candidates who wish to extend their post-16 study of mathematics can study A- or AS-level Further Mathematics, although numbers doing so have been dwindling. In the 1980s around 12 000 students were taking 'double mathematics' each year at A-level, whereas in the late 1990s this had levelled out at just over 5000 taking A-level Further Mathematics, and around 2000 taking AS-level Further Mathematics – despite the growth in the numbers within the A-level cohort. A major factor has been the problem of maintaining financially viable group sizes in schools and colleges. The Gatsby Foundation is supporting a project to make these courses more widely available to students through distance learning. We welcome such initiatives to encourage greater take up of these courses, as students taking such qualifications are much better placed for success in undergraduate studies in mathematics, physics and engineering. Similar arguments apply about making these courses more interesting and challenging by including both more geometry and the greater use of geometrical contexts.

There is now a range of post-16 qualifications called 'Free Standing Mathematics Units' (FSMUs) which are designed to support students in their other studies and which should enable more students to pursue mathematics post-GCSE. The FSMUs are available at three levels. There are some FSMUs which include geometry, but none at level 3, the advanced level. We recommend that a geometry unit be developed at level 3. In Autumn 2001 a new AS-level qualification called 'Use of Mathematics' is to be introduced, based on advanced level FSMU modules. The working group had understood that this new qualification was intended for students who would not otherwise take a post GCSE mathematics course, such as those specialising in the arts, humanities and social sciences. At the time of writing, the qualification is still in development but it appears that it will now be predominantly a course in mathematical modelling with no geometrical content at all. Thus there would still appear to be a gap in the market for an AS-level qualification in mathematics which will appeal to students specialising, say, in the arts and humanities and for whom geometry might be an attractive element of study.

The working group believes that there is still more to be done to ensure that there is a sufficient range of level 2 and 3 mathematics qualifications to attract greater numbers of students to continue their studies in mathematics post-16. We recommend that the range of level 3 mathematics qualifications be reviewed to ensure that students have the opportunity to study geometry further. In particular the cultural, aesthetic and historical aspects of geometry, such as the development of perspective, should be of considerable appeal to many of those students from the arts and humanities who currently drop mathematics.

The Royal Society website provides links to current specifications of post-16 mathematics qualifications.

We consider that Curriculum 2000 may have an adverse effect on mathematics, stemming partly from its complexity and rigidity. However as major changes to 16-19 education are currently being implemented, it is not the time to make any detailed recommendations with respect to specific mathematics qualifications. Thus we make the following general recommendations.

Recommendation 12:

We recommend that a fundamental review be made of all 16-19 mathematics provision. This should include considering how:

a) the structure and content of the current AS/A-level Mathematics and Further Mathematics specifications can better meet the needs of students and include a greater emphasis on geometry; and

b) other post-16 mathematics qualifications, such as Free Standing Mathematics Units and AS-level Use of Mathematics, can enable students to have the opportunity to continue their study of geometry.

Recommendation 13:

We recommend that in the 16-19 curriculum the key skill, 'Application of Number', be re-titled 'Application of Mathematics' and that the range of qualifying mathematical studies be broadened so that students continue their study of geometry.

8 The role of assessment

Key Principle 6: The assessment framework for the curriculum should be designed to ensure that the full range of students' geometrical knowledge, skills and understanding are given credit.

We do not believe that many of the geometrical objectives in Recommendation 3 can be adequately assessed within the current framework of timed tests and examinations. Indeed, the current assessment framework is one of the major reasons why important aspects of geometry, such as work in 3-D, geometrical reasoning and the use of ICT have not been given sufficient attention in classrooms. It is only natural that teachers concentrate on aspects of a curriculum which carry the greatest assessment weighting - especially within the current climate of target setting in schools.

A number of questions set in national tests and examinations which are apparently about geometry are in practice mainly exercises in algebra. We would like to see national tests and examinations incorporate questions which test geometrical reasoning and applications of geometry. If, as we suspect, there is little experience in doing so, then we recommend that the QCA commissions work to develop more appropriate forms of examination questions in geometry.

GCSE examinations in 2003 will contain a compulsory course work element. This will consist of two extended tasks each contributing 10% of the total marks. One of these has to be from Ma4 *Handling data*. The other is to demonstrate skills from *Using and applying mathematics* in the context of either Ma2 *Number and algebra* or Ma3 *Shape, space and measures*. The working group welcomes to some extent this extension of assessment techniques but queries the rationale used to make Ma4 compulsory and not Ma3. We consider that the opportunity afforded for extended work in Ma3 would be an effective way to ensure that some of our objectives for geometry teaching are more effectively fulfilled. We are concerned that teachers faced with a choice between Ma2 and Ma3 may reject geometry in favour, say, of algebra - perhaps because of their own subject confidence or because they judge the more algorithmic nature of some forms of algebraic enquiry to be a 'safer bet'. Thus we recommend that GCSE mathematics should include some compulsory course work in geometry.

The examinable course work element in the new AS/A-level mathematics course is almost entirely restricted to the applications, such as statistics and mechanics. If, as we recommend, greater opportunity is afforded to students to extend their study of geometry on these courses then a review of the appropriate means of assessment is also needed. By contrast the FSMUs have their own forms of assessment - usually 50% examination and 50% coursework. Many units specify and assess the use of appropriate ICT.

If the assessment framework for the curriculum in geometry can be developed to include both better examination questions and a reasonable contribution from extended course work, then we believe that teachers will also be encouraged to develop formative assessments in geometry for their students. It is a wider question than this working group's remit to consider whether teachers' assessments of students' progress should contribute to National Curriculum assessment and to public examinations. Such a change would require the kind of reinstatement of teachers' professional judgements which is discussed in the recent government Green Paper [DFEE, 2001].

Recommendation 14:

We recommend that a review be made of the methods of assessment and examination used in mathematics at Key Stage 3, at GCSE and in post-16 qualifications to ensure that appropriate credit is given for the attainment of specific geometrical objectives.

9 Teaching of geometry

Key Principle 7: The most significant contribution to improvements in geometry teaching will be made by the development of good models of pedagogy, supported by carefully designed activities and resources, which are disseminated effectively and coherently to and by teachers.

We did not enquire further into the impact of different forms of school organisation, such as 'setting' and mixed-ability teaching, nor did we reach a consensus about the issues arising from providing a more inclusive curriculum or from providing some more differentiated curricula. We anticipate that pilot studies in good practice may provide some helpful guidance in respect of these issues.

We now turn to the description of the 11-16 National Curriculum. The phraseology used throughout is *"pupils should be taught to…".* The National Curriculum handbook sets out in some detail what should be taught, but not why, or how. A good deal of scope for interpretation still rests with the teacher. We are aware that a tendency has recently developed in teaching the mathematics National Curriculum which breaks it down into a large number of very limited objectives - sometimes known as 'bite-sized chunks'. Such an approach can, and often does, result in fragmentation and in the failure to develop important links between curriculum areas. We believe that the successful implementation of the Ma3 component in the classroom will only be achieved if teaching programmes are focused and coherent, and if they develop links within geometry and mathematics generally where appropriate. In Appendix 13 we give some examples of ways in which aspects of the geometry curriculum could be integrated within a particular theme, and also where aspects of geometry could be linked with other areas of mathematics such as algebra and handling data.

Individual teachers implement the curriculum by planning schemes of work and lessons for their classes. So it is a matter of the greatest importance to ensure that teachers have the necessary information, skills and resources to interpret the aims and objectives of the curriculum. Recent moves to ameliorate problems of recruitment and retention of teachers, together with the government's intention to modernise working practices in health and education, mean that there is now a much more favourable climate for improving the system of teachers' continuing professional development (CPD). The working group welcomes the declared intention to provide CPD support to improve and update teachers' subject knowledge and related pedagogy.

Research, such as that reported by Hoyles and Healey, has confirmed the views of experienced teachers in schools that there are many teachers of mathematics who have large gaps in their knowledge of geometry. Similarly, we believe that there are also many teachers who have been taught geometry through styles of teaching which we would not advocate as appropriate. Thus our view is that in respect of geometry teaching there is a need for a significant CPD initiative. Government is giving greater attention to spreading good practice between teachers and schools through initiatives, such as the beacon schools. The DfES (whilst still DfEE) recently launched its CPD strategy. This has its own website at: www.dfee.gov.uk/teachers/cpd where CPD initiatives are presented under the strap line "*Learning from each other... Learning from what works*". The working group welcomes this approach. We regard it as vital that pilot studies should be carried out without delay to identify and enhance good practice in the teaching of geometry. At the same time planning should take place for a national system of provision for CPD in geometry and its teaching. This could be within the framework of the Key Stage 3 strategy. One idea which has received some support is the provision of two week geometry summer schools for serving teachers, teachers currently in training and those about to embark on a course of initial teacher education (ITE). Financial inducements may be needed to encourage attendance in vacation time. By concentrating on subject knowledge in geometry, as well as its teaching, such courses should generate enjoyment of mathematics and thus help as one part of the long term process of sustaining or renewing teachers' enthusiasm for it.

New graduates entering courses of initial teacher education have very varied backgrounds in geometry. Many will have experienced little, if any, geometry at sixth form or university level. Within the current statutory curriculum for the initial training of secondary mathematics teachers there is little scope to provide the rich overview of geometry that we believe is essential for effective teaching. Further professional development for teachers early in their career is essential, but is most likely to concentrate on the development of their teaching skills. So it is important that, in parallel with developments in CPD to support the teaching of geometry, there is a recognition of the need to improve the geometrical background of those intending to enter mathematics teaching during, or before, their initial training.

In order to support the developments in the effective teaching of geometry which we seek there is a need for a variety of materials in both printed and digital form, as well as resources such as models, posters, activity kits, videos, libraries of digital images, computer software and the like. Some of this already exists and we provide a far from exhaustive list of these in Appendix 14. An important activity will be to review the current provision and to develop new materials and resources as appropriate.

Geometry teaching outside primary schools can be, and has been, conducted with a minimal amount of equipment - such as a stick of chalk and a piece of string (echoing images of ancient Greeks drawing in the sand). Our view is that teachers should now have at their disposal an appropriate variety of equipment from which to select, depending on fitness for purpose. In particular we wish to see the potential of ICT realised in supporting the teaching and learning of geometry. There is already software available, such as for dynamic geometry (DGS), but its use is not widespread. Many schools do not have licences for the software. There is also a need for the development of additional software, such as to support work in 3-dimensions. Increasing numbers of schools and colleges are now being equipped with interactive whiteboards - where a computer image is projected onto a touch sensitive screen. This medium has considerable potential for interactive whole-class teaching of geometry. We would like to see the funding to schools for ICT being used more effectively to support the geometry curriculum.

The mathematics professional associations have a key role to play in each of these developments in partnership with the newly formed General Teaching Council (GTC) and other bodies such as the Royal Society, Higher Education institutions, the National Numeracy Strategy, QCA, Ofsted, TTA and BECTa.

Recommendation 15:

We recommend that the relevant government agencies work together with bodies, such as the mathematics professional associations represented on JMC, to provide a coherent framework for supporting the development of teaching and learning in geometry. This will involve:

a) the recognition and development of good practice in geometry teaching through pilot studies and research;
b) the design of programmes of continuing professional development and initial teacher education;
c) the production of supporting materials; and
d) the establishment of mechanisms to provide supporting resources, including ICT.

10 Improving the take up of mathematics

Key Principle 8: It is a matter of national importance that as many of our students as possible fully develop their mathematical potential. Geometry, with its distinctive appeal, should make mathematics attractive to a wider range of students.

In launching UK Maths Year 2000, the Prime Minister made clear the importance of mathematics in the education of those creative and flexible thinkers on whom our national economic prosperity will depend. The demands of commerce and industry for articulate graduates with mathematical skills far outstrips the supply of graduates in mathematics and related fields. One consequence of this is the current severe shortage of new teachers, especially for mathematics in secondary schools. For a variety of reasons, insufficient numbers of our ablest students are choosing to pursue mathematics as a specialism following both GCSE and AS/A-level. There is no universal panacea. So it is vital that we take any opportunity to review where and how the subject could be made more interesting, attractive, relevant, challenging, rewarding and engaging to all students. We are convinced that geometry has a lot to offer in this respect. For some students it may be the logical aspects which are the most appealing, for others it may be the visualising, or the modelling, or the historical and cultural, or the visual and aesthetic aspects.

In general we believe that students are not given enough information about the importance of mathematics in the world of work, and the significant advantages a mathematical education can bestow in terms of employability. The ways in which the performance of secondary schools and colleges are published through examination results takes no account of the relative national economic importance of some subjects over others. So, for example, there is a positive incentive for institutions to persuade students to choose subjects in which it is easier to achieve high grades at A-level than those subjects judged harder, which include mathematics and physics.

Overall the profile of mathematics needs to be higher in schools, colleges and universities if we are to attract more students at all levels of attainment to realise their potential in the subject. This means that still more needs to be done to improve the status of mathematics teaching, and to attract (and retain) good recruits. It also means that students should be made more aware of the relationships between mathematics and the other subjects they study. We believe that geometry is a good vehicle for achieving this aim.

Recommendation 16:

We recommend, in terms of mathematics in general, that:

a) better publicity and information be provided to schools, students and parents about the career opportunities afforded by studying mathematics; and

b) ways be sought to encourage schools and colleges to attract more students to study mathematics post-16, particularly at A-level.

11 Conclusion

For mathematics 11-16, we have concluded that the geometrical content of the new National Curriculum, with a few adjustments, forms an appropriate basis for a good geometry education. In order for this to be achieved, considerable changes are needed in the way geometry is taught. It is vital that those working to improve mathematics education ensure that their work contributes significantly to improvements in geometry (as well as mathematics) teaching. Bringing about improvements in geometry teaching will require a significant commitment to a substantial programme of continuing professional development, together with the development of appropriate supporting materials.

For mathematics post-16 we have concluded that there are insufficient opportunities for students to build on their 11-16 studies in geometry. Those concerned with curriculum design need to review the structure of post-16 qualifications in mathematics to ensure they provide better opportunities for students to continue to study geometry. More generally there is at present a severe shortage of those with good mathematical skills - and the provision of challenging and interesting geometrical content and contexts should be a valuable means to make mathematics a more attractive subject of study for more students.

12 References and glossary

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Glossary

AS level	Advanced Subsidiary Level – a qualification between GCSE and A-level
BECTa	British Educational and Communications Technology Agency
CADCAM	Computer Aided Design and Computer Aided Manufacture
CPD	Continuing Professional Development
DfEE	Department for Education and Employment (now replaced by the DfES)
DfES	Department for Education and Skills
DGS	Dynamic Geometry Software
FSMU	Free Standing Mathematics Unit
GCSE	General Certificate of Secondary Education
GTC	General Teaching Council
HE	Higher Education
ICMI	International Commission on Mathematics Instruction
ICT	Information and Communications Technology
IT	Information Technology
ITE	Initial Teacher Education
ITT	Initial Teacher Training
JMC	Joint Mathematical Council of the United Kingdom
KS	Key Stage (of the National Curriculum)
LEA	Local Education Authority
Ma3	"*Shape, space and measures*" component of the mathematics National Curriculum
NC	National Curriculum
NNS	National Numeracy Strategy
NOF	New Opportunities Fund (a Government funding initiative)
Ofsted	Office for Standards in Education
QCA	Qualifications and Curriculum Authority

QTS	Qualified Teacher Status
SEU	Standards and Effectiveness Unit (of the DfES / DfEE)
TIMSS	Third International Mathematics and Science Study
TTA	Teacher Training Agency