Can you hear the shape of a graph?

Project executives

Director: Dr Ram Band
Deputy Director: Adam Sawicki
Consultant: Dr Dane Comerford

Posters and Exhibit Layout

- **Quantum Graphs**: “I am a quantum graph - how do I sound?”
- **Inverse Problems**: “Here is my sound - how do I look?”
- **Nodal Domains**: “Counting Vibrations - puzzling patterns”

Team Leader: Joanna Hutchinson
Team members: Orestis Georgiou, Maxim Kirsebom, Peter Shadbolt and Nick Simm.

Exhibit Items

**Multi-touch table**
Visitors can construct their own *quantum graphs* on a multi-touch surface! The behaviour of each graph is then calculated and visualized on the table and wall screens in real-time. This high-tech exhibit complements the ‘Vibrating graph’ exhibit by calculating and exposing extra information about the system in question, such as the spectrum, the sound of the graph, vertex conditions, etc. The multi-touch capability of this exhibit allows visitors to work together to explore ideas such as isospectrality and inverse problems.

Team leader: Peter Shadbolt.
**Vibrating graph**
Enthusiastic visitors are invited to connect together any number of elastic strings to form a graph. Switch on the driving vibrator, and tune the frequency so that the graph resonates. Nodal domains are then clearly distinguishable. By increasing the frequency, you can achieve several fascinating vibration patterns! This is a ‘hands on’ activity, appealing to the practically minded visitors.

Team leader: Oliver Sargent
Team members: Orestis Georgiou and Ben Leadbetter.

**Chladni Plates and 3D Wave Machine**
These exhibit items demonstrate the concept of spectrum and nodal **domains** in different dimensions. You can create beautiful patterns both on the vibrating Chladni plates and on the rotating string. Discover the different parameters which control the shape and the number of the appealing nodal domains.

Team leader: Joanna Hutchinson
Team members: Orestis Georgiou, Maxim Kirsebom and Nick Simm.

**Arts and crafts corner**
- Visitors can draw, cut out and fold their own pair of isospectral drums or graphs.
- Experiment with vibrating slinkys to see how many different nodal domains you can produce using one slinky or joining many slinkys together to form a graph. Enthusiastic visitors are awarded a slinky to take home!

Team leader: Joanna Hutchinson
Team members: Orestis Georgiou and Nick Simm.

**Theoretic material**
Three leaflets and one fact sheet are distributed during the exhibition. The leaflets content is associated to the exhibit posters. Two of the leaflets are shaped as isospectral drums and have isospectral graphs drawn on their reverse. The fact sheet gives a more thorough explanation about quantum graphs, inverse problems and isospectrality.

Team leaders: Joanna Hutchinson (design) and Lionel Kameni (content)
Team members: George Adje, Yating Deng, Jasmin Meinecke, Andy Poulton, Peter Shadbolt and Nick Simm.

**Websites**
http://www.maths.bristol.ac.uk/~maxrb
Can you hear the shape of a graph?

I am a quantum graph - how do I sound?

You can draw a quantum graph by placing any number of points on a page and connecting them with lines. Points on the graph are called vertices and the lines are known as edges. A quantum graph as a physical object would look like a collection of guitar strings, tied together. When you pluck any of the strings the vibration carries along the whole graph. The network of strings can produce an infinite number of notes when plucked in different ways. For each note, the string vibrates at a particular frequency. The collection of all frequencies is known as the spectrum of the graph.

Visualisations of a five-edged quantum graph producing its 8th, 13th and 16th notes

This is my sound - how do I look?

You are awoken from your slumber by hearing your neighbour playing a loud instrument. Can you tell which instrument disturbed your sleep just by hearing the sound it makes? Mathematicians solve problems like this when the instrument is actually a quantum graph. Could you guess what your neighbour’s graph looks like just by hearing its sounds? This is an example of what scientists call an inverse problem.

The strategy for answering this question involves listening to the pitch of the notes. Long strings have low notes, with a low vibration frequency; we can work out a quantum graph’s total length from its sound spectrum. Visualising how the individual strings are connected, the shape of the graph, is more complicated.

The shape of a graph is related to its periodic orbits. These are journeys along the edges of the graph which start and finish at the same vertex.

A few periodic orbits of a graph

A special formula shows how the spectrum of sounds determines the lengths of all periodic orbits of a graph [see reference 1 below]. For most graphs, this set of lengths, determined by their spectrum, defines their shape [reference 2]. However, there are pairs of graphs with different shapes but a common spectrum. These are isospectral graphs.
Graphs or drums which sound the same

For most of the 20th century, mathematicians and physicists did not know whether isospectral graphs existed. They started investigating this question for another musical instrument – the drum. For Mark Kac, a mathematician, a drum does not have to be circular, but can have any shape. He was curious to find out whether it is possible to build sets of isospectral drums – drums which look different, but sound exactly the same [reference 3]. Since 1966, when Kac started his exploration, this question bothered many mathematicians and physicists. The answer was only found in 1992, when the three mathematicians, Carolyn Gordon, David Webb and Scott Wolpert discovered the first example of isospectral drums [reference 4]. Gordon et al. used a formal theory derived by Toshikazu Sunada to produce their pair of drums [reference 5].

More recently, Ram Band, Ori Parzanchevski and Gilad Ben-Shach developed a new method enabling the construction of isospectral drums and graphs [references 6 and 7].

An application of the isospectral construction

Picture a square-shaped quantum graph, together with its four axes of reflective symmetry [labeled x, y, u, v]. Using these symmetries we can create a pair of isospectral graphs, by ‘desymmetrising’, or cropping, the graph. One is produced if we cut the ‘parent’ graph along its axes, x and y, and keep the remaining corner piece. A second graph is created by cutting along axes u and v and taking the side section that contains a loop. These two new graphs have been produced by this new method for general isospectral construction. We can redraw the pair of graphs with no change in their properties.

The isospectral construction method begins with a graph with certain symmetry properties. This ‘parent’ graph can be desymmetrised in a number of ways to generate new graphs. They are not necessarily isospectral and we need to follow a certain rule to obtain isospectrality. The recently developed isospectral theorem supplies us with this rule [references 6 and 7].

Quantum graphs - applications

Quantum graphs, both simple and complex, form a sub-domain within the general research of networks. This research ranges from the human arterial and neural systems to the World Wide Web. We use quantum graphs today to simplify complicated phenomena. They acquired the name “quantum graphs” because one of their earliest applications was to explain molecules – the ubiquitous inhabitants of the quantum mechanical world. They are increasingly useful for explaining other phenomena from “quantum chaos” to understanding the propagation of electromagnetic and acoustic waves used in modern communications and medical technologies [references 8 and 9].


Most of the up to date research from this exhibit was conducted during the PhD of Ram Band while supervised by Professor Uzy Smilansky in the Weizmann Institute of Science, Israel. Further reading: http://www.maths.bristol.ac.uk/~maxrb
Imagine a network of strings connected together, as in the picture above. This is an example of what physicists call a quantum graph. Quantum graphs, like the single string, also have a spectrum of harmonics associated with them. These harmonics are studied by University of Bristol mathematicians and physicists.

Picture a single guitar string. It can produce an infinite number of different notes when plucked in different ways. These are called harmonics. For each harmonic, the string vibrates at a particular frequency. This collection of frequencies is known as the spectrum of the string.

Use our exhibit to build your own quantum graph. Be creative! Vary the number and length of the strings, and try different ways to connect them. Each time, you will get a new quantum graph with its own spectrum of sounds; how does this spectrum change with the quantum graph?

The harmonics of a single guitar string
Imagine that your friend decides to build a quantum graph while on holiday somewhere far away. They find out the quantum graph's harmonics and send you the list of frequencies on a postcard. Can you rebuild your friend's quantum graph just by knowing this list? This riddle is an example of an inverse problem.

We can find different quantum graphs which share the same spectrum. These quantum graphs sound the same and are called isospectral. Turn over this leaflet and see a quantum graph; find its isospectral partner in our exhibit. Listen out for other quantum graphs that sound the same, try these two for example...

Real world examples of inverse problems include radar, sonar and ultrasound. This is when indirect measurements help us see the shape of an object.

Can every quantum graph be rebuilt just from knowing its spectrum of sounds? That is, can you hear the shape of a graph?

Shape of leaflet inspired by figure from "Some planar isospectral domains" by Buser et al. in Int. Math. Res. Not. 9, 391-400 (1994)

Use of ultrasound
Use of radar

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There is no sound without vibration. When you hear a sound, you know that something is vibrating.

Ernst Chladni, a German physicist and musician was interested in the visual effects of vibration. In the late 18th century, Chladni was traveling around Europe captivating audiences with his demonstrations. He poured sand on a brass plate and made it vibrate using a violin bow. Depending on the frequency of vibrations produced, he saw the sand arrange itself in various beautiful patterns which astounded the people who came to see his lectures.

The sand gathered on special lines on the plate which did not vibrate at all. In between these lines were vibrating regions, clear of sand, called nodal domains.

Similarly, when a quantum graph vibrates a few special points will remain still. In between those points are the vibrating regions. These are the nodal domains of a quantum graph. For each harmonic in the spectrum of a quantum graph you can count the number of nodal domains - this sequence is known as the nodal count.

If your quantum graph consisted of a single string, its nodal count would simply be 1, 2, 3, 4, ...

Explore our exhibit: try to build a quantum graph which has a more interesting nodal count. The sequence of numbers will depend on how you connect your graph.

Is there a rule that determines the nodal count of a quantum graph from its shape? If you are told the nodal count of a quantum graph, could you reconstruct it? The answers to these questions are still not fully understood. University of Bristol mathematicians are working on these problems now.
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www.maths.bristol.ac.uk

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